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Toric ideals and nonregular triangulations of convex polytopes

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整凸多面体の正則三角形分割は，グレブナー基底の理論を用いて解析できることが知られている．このような事情もあって，正則三角形分割については様々な結果が知られているが，非正則三角形分割については，未だその基礎理論が整備されておらず，有用な結果はほとんど知られていないように思われる．当該論文では，正則三角形分割に関する結果を非正則三角形分割に一般化し，非正則三角形分割の代数的基礎理論の樹立，つまり，グレブナー基底とイニシャルイデアルの理論の三角形分割の枠組においての一般化を行なった．

第一に，非正則三角形分割の代数的特徴付けを行なった．そのために，トーリックイデアルのサーキットと呼ばれるものに注目した．まず，既知であった，正則三角形分割のStanley–Reisner イデアルと凸多面体に対応するトーリックイデアルのイニシャルイデアルの根基イデアルの一対一対応において，トーリックイデアル全体ではなく，サーキット集合のみのイニシャルイデアルを考えればよいことを示した．さらに，イデアルのサーキット全体に対して，“マーキング” を考えた．マーキングとは，サーキットの各二項式に対して，そのいずれかの項を指定する操作をいう．特に，項順序は自然にマーキングを定義する．このとき，任意の三角形分割に対してあるマーキングが存在し，三角形分割のStanley–Reisner イデアルがマーキングによって定義される単項式イデアルの根基イデアルと一致することを示した．また，サーキット上のマーキングは一般には三角形分割に対応しないが，与えられたマーキングが三角形分割に対応するための必要十分条件を求めた．

第二に，凸多面体の三角形分割のフリップと呼ばれる操作の代数的特徴付けを行なった．フリップとは凸多角形の三角形分割に対する対角変形を一般次元に拡張したものである．正則三角形分割のフリップについては多くの事が知られているが，非正則三角形分割のフリップについては有用な結果はあまり知られていないように思われる．そこで，フリップで移りあう2つ三角形分割に対して，それぞれの三角形分割に対応する適当な2つのマーキングが存在し，サポートをもつサーキットのマーキングのみが異なり，それ以外はマーキングが一致するようにできるということを示した．また，与えられた三角形分割がどのサーキットにサポートをもつかを判定する必要十分条件も求めた．この判定条件にはグレブナー基底に対する Buchberger の判定法に類似する点が見られる．

第三に，以上の結果を有限グラフに付随する凸多面体の三角形分割に関する問題に応用した．性質「正則三角形分割から1回のフリップで得られる非正則単模三角形分割をもつが，正則単模三角形分割は持たない」をみたす凸多面体は，[O–H_1]で構成した辺凸多面体が唯一のものであったが，この辺凸多面体から全く同じ性質をもつ辺凸多面体の無限系列の構成を行なった．

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Introduction

Let $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ be a finite subset of $\mathbb{Z}^d$ and suppose that $\mathcal{A}$ is contained in a hyperplane which does not contain the origin. We may assume that $rank(\mathcal{A}) = d$ if we regard $\mathcal{A}$ as a matrix. (If not, we can delete suitable rows of $\mathcal{A}$.) Let K be a field and $K[\mathbf{t}, \mathbf{t}^{-1}] = K[t_1, t_1^{-1}, \ldots, t_d, t_d^{-1}]$ the Laurent polynomial ring in d variables over K. Then, we write $K[\mathcal{A}]$ for the subalgebra of $K[\mathbf{t}, \mathbf{t}^{-1}]$ which is generated by $\mathbf{t}^{\mathbf{a}_1}, \ldots, \mathbf{t}^{\mathbf{a}_n}$ over K. Here $\mathbf{t}^{\mathbf{a}_i} = \prod_{j=1}^{d} t_j^{\alpha_j}$ if $\mathbf{a}_i = (\alpha_1, \ldots, \alpha_d)$. Let $K[\mathbf{x}] = K[x_1, x_2, \ldots, x_n]$ denote the polynomial ring in n variables over K and $\pi : K[\mathbf{x}] \to K[\mathcal{A}]$ the surjective homomorphism of semigroup rings defined by $\pi(x_i) = \mathbf{t}^{\mathbf{a}_i}$ for all $1 \le i \le n$. We write $I_\mathcal{A}$ for the kernel of π and call $I_\mathcal{A}$ the *toric ideal* associated with the affine semigroup ring $K[\mathcal{A}]$.

Let $\mathcal{P}_\mathcal{A}$ be the convex hull of $\mathcal{A}$. A *triangulation* Δ of $\mathcal{P}_\mathcal{A}$ is a set of subsimplices of $\mathcal{P}_\mathcal{A}$ which satisfies the following conditions:

(i) All vertices of each $\sigma \in \Delta$ belong to $\mathcal{A}$;

(ii) $\mathcal{P}_\mathcal{A} = \bigcup_{\sigma \in \Delta} \sigma$;

(iii) If F is a face of $\sigma \in \Delta$, then $F \in \Delta$;

(iv) If $\sigma_1, \sigma_2 \in \Delta$, then $\sigma_1 \cap \sigma_2$ is a face of both σ_1 and σ_2.

One of the ways to represent a triangulation Δ of $\mathcal{P}_\mathcal{A}$ is to consider the Stanley–Reisner ideal $I_\Delta = (x_{i_1} \cdots x_{i_r} \; ; \; \{\mathbf{a}_{i_1}, \ldots, \mathbf{a}_{i_r}\} \notin \Delta) \subset K[\mathbf{x}]$ of Δ. Here, we write $\{\mathbf{a}_{i_1}, \ldots, \mathbf{a}_{i_r}\}$ for the polytope whose vertices are $\{\mathbf{a}_{i_1}, \ldots, \mathbf{a}_{i_r}\}$. If Δ is a "regular" triangulation [G–K–Z], then an algebraic approach is known as follows:

Proposition 0.1 ([Stu, Theorem 8.3]). *Let $\prec$ be a term order on $K[\mathbf{x}]$. Then, the radical ideal $\sqrt{in_\prec(I_\mathcal{A})}$ of the initial ideal $in_\prec(I_\mathcal{A})$ of $I_\mathcal{A}$ is the Stanley–Reisner ideal of a triangulation of $\mathcal{P}_\mathcal{A}$.*

Triangulations discussed in Proposition 0.1 are called *regular* (or *coherent*). The main purpose of this paper is to generalize Proposition 0.1 to nonregular triangulations.

In Section 1, we study basic results on markings and circuits of toric ideals. A binomial $f \in I_\mathcal{A}$ is called *circuit* if f is irreducible and has minimal support. Here, the support of $f = \prod_{i=1}^{n} x_i^{p_i} - \prod_{j=1}^{n} x_j^{q_j}$ is defined by $supp(f) = \{x_i \; ; \; p_i > 0 \text{ or } q_i > 0\}$. Let $C_\mathcal{A}$ be the set of all circuits of $I_\mathcal{A}$. In order to give an algebraic approach to nonregular triangulations, we consider a marking $in(\cdot)$ on $C_\mathcal{A}$, i.e., for each binomial $f \in C_\mathcal{A}$, $in(f)$ is one of the terms of f. We say that a marking $in(\cdot)$ is *coherent* if there exists a term order $\prec$ such that $in(f) = in_\prec(f)$ for all $f \in C_\mathcal{A}$. Let $in(C_\mathcal{A})$ denote the monomial ideal $(in(f); f \in C_\mathcal{A})$. Proposition 1.4 says that $\sqrt{in_\prec(I_\mathcal{A})} = \sqrt{in_\prec(C_\mathcal{A})}$ holds for any term order $\prec$. Hence, it turns out that, in Proposition 0.1, it is sufficient to consider only the set of circuits $C_\mathcal{A}$ instead of $I_\mathcal{A}$. Note that $in_\prec(I_\mathcal{A}) \neq in_\prec(C_\mathcal{A})$ in general. See Example 1.5.

In Section 2, we study the relation between triangulations and markings on $C_\mathcal{A}$. Theorem 2.1 guarantees that, given a triangulation Δ of $\mathcal{P}_\mathcal{A}$, there exists a marking

$in(\cdot)$ on $C_{\mathcal{A}}$ such that $I_\Delta = \sqrt{in(C_{\mathcal{A}})}$. Thanks to Proposition 1.4, Theorem 2.1 generalizes Proposition 0.1. On the other hand, the converse of Theorem 2.1 is false in general, i.e., there is a marking $in(\cdot)$ such that $\sqrt{in(C_{\mathcal{A}})}$ is not the Stanley–Reisner ideal of any triangulation of $\mathcal{P}_{\mathcal{A}}$. See Example 2.2. However, we show that every marking $in(\cdot)$ can be associated with the simplicial complex Δ_{in} on the vertex set $\mathcal{A}$. See Proposition 2.3. If a marking $in(\cdot)$ satisfies that $\sqrt{in(C_{\mathcal{A}})}$ is the Stanley–Reisner ideal of a triangulation of $\mathcal{P}_{\mathcal{A}}$, then we call $in(\cdot)$ a *geometric marking* on $C_{\mathcal{A}}$. We study a criterion for markings on $C_{\mathcal{A}}$ to be geometric markings. See Theorem 2.6.

In Section 3, we discuss flips of triangulations of $\mathcal{P}_{\mathcal{A}}$. We say that a triangulation Δ of $\mathcal{P}_{\mathcal{A}}$ is *supported* on a circuit $f = \mathbf{x}^{u^+} - \mathbf{x}^{u^-} \in C_{\mathcal{A}}$ if the following two conditions are satisfied:

(i) $\prod_{i \in supp(f)\setminus\{j\}} x_i \notin I_\Delta$ for all $j \in supp(\mathbf{x}^{u^+})$;
(ii) For all monomials $m \in K[\mathbf{x}]$ such that $supp(m) \cap supp(\mathbf{x}^{u^+}) = \emptyset$ and for all $j_1, j_2 \in supp(\mathbf{x}^{u^+})$, we have $m \cdot \prod_{i \in supp(f)\setminus\{j_1\}} x_i$ belongs to I_Δ if and only if $m \cdot \prod_{i \in supp(f)\setminus\{j_2\}} x_i$ belongs to I_Δ.

If a triangulation Δ of $\mathcal{P}_{\mathcal{A}}$ is supported on a circuit $f \in C_{\mathcal{A}}$, then we can construct a new triangulation by taking away all the simplices of the form $supp(m) \cup supp(f) \setminus \{i\}$ where $i \in supp(\mathbf{x}^{u^+})$ and $m \in K[\mathbf{x}]$ is a monomial with $supp(m) \cap supp(f) = \emptyset$ and adding the simplices of the form $supp(m) \cup supp(f) \setminus \{j\}$ where $j \in supp(\mathbf{x}^{u^-})$ and the same m. We call this operation a *flip* (or *modification* or *bistellar operation*) along f. It is known [G–K–Z] that any two regular triangulations are connected by finite flips. Using this fact, there are several algorithms which enumerate all regular triangulations of a convex polytope. Consult [De], [G–K–Z] and [Rei] for the details about flips. On the other hand, recently, it turns out [San] that there exists a convex polytope having a (nonregular) triangulation which is supported on NO circuit. In Theorem 3.3 and Theorem 3.4, we give an algebraic approach to flips.

A configuration $\mathcal{A}$ is called *unimodular* if all triangulations of $\mathcal{P}_{\mathcal{A}}$ are unimodular. It is known that $\mathcal{A}$ is unimodular if and only if both terms of any circuits of $I_{\mathcal{A}}$ are squarefree. Hence, $\sqrt{in(C_{\mathcal{A}})} = in(C_{\mathcal{A}})$ if $\mathcal{A}$ is unimodular. Moreover, [Stu, Proposition 8.11] says that, if $\mathcal{A}$ is unimodular, then $C_{\mathcal{A}}$ equals to the universal Gröbner basis $\mathcal{U}_{\mathcal{A}}$, i.e., $C_{\mathcal{A}}$ is a Gröbner basis with respect to every term order. In Theorem 2.6, we state that $in(\cdot)$ is a geometric marking if and only if every monomial in $K[\mathbf{x}]$ can reduce to a monomial which does not belong to $in(C_{\mathcal{A}})$ with respect to $in(\cdot)$ modulo a subset of $C_{\mathcal{A}}$ by a suitable reduction if $\mathcal{A}$ is unimodular. On the other hand, Corollary 3.5 guarantees that we know which circuits support a triangulation if we compute S-polynomials of a subset of $C_{\mathcal{A}}$ in unimodular case.

In Section 4, we apply these results to problems of polytopes arising from finite graphs. Recently, the following six properties on a configuration $\mathcal{A}$ have been investigated by many papers on commutative algebra and combinatorics:

(i) $\mathcal{A}$ is unimodular;
(ii) $\mathcal{P}_{\mathcal{A}}$ is compressed, i.e., the regular triangulation with respect to any reverse

lexicographic order is unimodular;

(iii) $\mathcal{P}_\mathcal{A}$ possesses a unimodular regular triangulation;

(iv) $\mathcal{P}_\mathcal{A}$ possesses a unimodular triangulation;

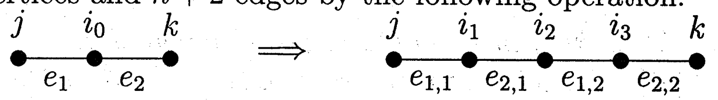
(v) $\mathcal{P}_\mathcal{A}$ possesses a unimodular covering;

(vi) $\mathcal{P}_\mathcal{A}$ is normal i.e., the semigroup ring $K[\mathcal{A}]$ is normal.

There is the hierarchy (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (vi). However, the converse of each of the five implications is false.

Let G be a finite connected graph having no loop and no multiple edge on the vertex set $V(G) = \{1, 2, \ldots, d\}$ and $E(G) = \{e_1, e_2, \ldots, e_n\}$ the set of edges of G. If $e = \{i, j\}$ is an edge of G joining $i \in V(G)$ with $j \in V(G)$, then we define $\rho(e) \in \mathbb{R}^d$ by $\rho(e) = \mathbf{e}_i + \mathbf{e}_j$. Here $\mathbf{e}_i$ is the i-th unit coordinate vector in $\mathbb{R}^d$. Let $\mathcal{A}_G = \{\rho(e) \ ; \ e \in E(G)\}$. We set $\mathcal{P}_G$ for $\mathcal{P}_{\mathcal{A}_G}$ and call $\mathcal{P}_G$ the *edge polytope* of G. We set $K[G]$ for $K[\mathcal{A}_G]$ and call $K[G]$ the *edge ring* of G and set I_G for $I_{\mathcal{A}_G}$ and call I_G the *toric ideal* of G. See also [O–H$_1$], [O–H$_2$], [O–H$_3$] and [O–H$_4$].

Suppose that G has a vertex i_0 of degree 2. Then, we can construct a new graph $\widehat{G}$ with $d+2$ vertices and $n+2$ edges by the following operation:



By using the results in section 1–3, we can define a bijection $\bar{\psi}$ from the set of all triangulations of $\mathcal{P}_G$ to the set of all triangulations of $\mathcal{P}_{\widehat{G}}$ which preserves regularity, unimodularity and flip connectivity. See Theorem 4.7 and Theorem 4.8.

In [O–H$_1$], an edge polytope none of whose regular triangulations is unimodular and having a unimodular triangulation obtained by one flip from a regular triangulation was studied. However, any other polytope which has the same property seems to be not known so far. In this paper, we give an infinite family of edge polytopes which have the same property. From $\mathcal{P}_{G_1}$ in Example 4.1, we get an infinite family of normal edge polytopes having the same property as $\mathcal{P}_{G_1}$ since G_1 has five vertices $\{v_1, v_2, \ldots, v_5\}$ of degree 2. Let $G_{(p_1,p_2,\ldots,p_5)}$ be the graph obtained from G_1 by applying the above operation $p_i - 1$ times to the vertex v_i for $1 \le i \le 5$. $G_{(p_1,p_2,\ldots,p_5)}$ has $2\sum_{i=1}^5 p_i$ vertices and $5 + 2\sum_{i=1}^5 p_i$ edges. Thanks to Theorem 4.7 and Theorem 4.8, we can show that the edge polytope $\mathcal{P}_{G_{(p_1,p_2,\ldots,p_5)}}$ is a normal (0,1)-polytope none of whose regular triangulations is unimodular and having a unimodular triangulation obtained by one flip from a regular triangulation of $\mathcal{P}_{G_{(p_1,p_2,\ldots,p_5)}}$. See Theorem 4.9.

1 Markings and circuits

In this section, we study markings and circuits of toric ideals. Let $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ be a finite subset of $\mathbb{Z}^d$ and suppose that $\mathcal{A}$ is contained in a hyperplane which does not contain the origin. Let K be a field and $K[\mathbf{t}, \mathbf{t}^{-1}] = K[t_1, t_1^{-1}, \ldots, t_d, t_d^{-1}]$ the Laurent polynomial ring in d variables over K. Then, we write $K[\mathcal{A}]$ for the subalgebra of $K[\mathbf{t}, \mathbf{t}^{-1}]$ which is generated by $\mathbf{t}^{\mathbf{a}_1}, \ldots, \mathbf{t}^{\mathbf{a}_n}$ over K. Here $\mathbf{t}^{\mathbf{a}_i} = \prod_{j=1}^d t_j^{\alpha_j}$

if $\mathbf{a}_i = (\alpha_1, \dots, \alpha_d)$. Let $K[\mathbf{x}] = K[x_1, x_2, \dots, x_n]$ denote the polynomial ring in n variables over K and $\pi : K[\mathbf{x}] \to K[\mathcal{A}]$ the surjective homomorphism of semigroup rings defined by $\pi(x_i) = \mathbf{t}^{\mathbf{a}_i}$ for all $1 \leq i \leq n$. We write $I_{\mathcal{A}}$ for the kernel of π and call $I_{\mathcal{A}}$ the *toric ideal* associated with the affine semigroup ring $K[\mathcal{A}]$. It is known that $I_{\mathcal{A}}$ is generated by homogeneous binomials. A binomial $f \in I_{\mathcal{A}}$ is called *circuit* if f is irreducible and has minimal support. Let $C_{\mathcal{A}}$ be the set of all circuits of $I_{\mathcal{A}}$.

Now, we consider the marking $in(\cdot)$ on $C_{\mathcal{A}}$, i.e., for each binomial $f \in C_{\mathcal{A}}$, $in(f)$ is one of the terms of f. We say that a marking $in(\cdot)$ is coherent if there exists a term order $\prec$ such that $in(f) = in_{\prec}(f)$ for all $f \in C_{\mathcal{A}}$. Let $in(C_{\mathcal{A}})$ denote the monomial ideal $(in(f); f \in C_{\mathcal{A}})$. It is known [Stu, Theorem 3.12] that

Proposition 1.1. *A marking $in(\cdot)$ on $C_{\mathcal{A}}$ is coherent if and only if every sequence of reductions modulo $C_{\mathcal{A}}$ with respect to $in(\cdot)$ terminates.*

For a positive integer p, there are only finite monomials of degree p in $K[\mathbf{x}]$. Since $C_{\mathcal{A}}$ consists of homogeneous binomials, we immediately have the following:

Corollary 1.2. *A marking $in(\cdot)$ on $C_{\mathcal{A}}$ is not coherent if and only if there exists a monomial $M \in in(C_{\mathcal{A}})$ such that there exists a sequence of reductions modulo $C_{\mathcal{A}}$ from M to M with respect to $in(\cdot)$.*

Let $f = \mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-} \in K[\mathbf{x}]$. Then, we associate f with a vector $\mathbf{u} = \mathbf{u}^+ - \mathbf{u}^- \in \mathbb{Z}^n$. Note that $f \in I_{\mathcal{A}}$ if and only if $\mathcal{A}\mathbf{u} = \mathbf{0}$. Here, we regard $\mathcal{A}$ as a matrix.

Lemma 1.3. *Let $\prec$ be a term order on $K[\mathbf{x}]$. Let $f = \mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-} \in I_{\mathcal{A}}$ with $in_{\prec}(f) = \mathbf{x}^{\mathbf{u}^+}$. Then, there exists a circuit $g = \mathbf{x}^{\mathbf{v}^+} - \mathbf{x}^{\mathbf{v}^-} \in C_{\mathcal{A}}$ such that $supp(\mathbf{v}^+) \subset supp(\mathbf{u}^+)$ and $supp(\mathbf{v}^-) \subset supp(\mathbf{u}^-)$ with $in_{\prec}(g) = \mathbf{x}^{\mathbf{v}^+}$.*

Proof. Factoring out common variables of $\mathbf{x}^{\mathbf{u}^+}$ and $\mathbf{x}^{\mathbf{u}^-}$, we may assume that $\mathbf{x}^{\mathbf{u}^+}$ and $\mathbf{x}^{\mathbf{u}^-}$ are relatively prime. Let ℓ be the minimum number of the cardinality of support of the binomial in $I_{\mathcal{A}}$. Let r be the cardinality of the support of $\mathbf{u}$.

If $r = \ell$, then there exists a circuit $g = \mathbf{x}^{\mathbf{v}^+} - \mathbf{x}^{\mathbf{v}^-} \in C_{\mathcal{A}}$ such that $supp(\mathbf{v}) = supp(\mathbf{u})$. By changing the sign of $\mathbf{v}$, we may assume that at least one of u_i/v_i is positive. Let $\lambda = \min(u_i/v_i > 0 \; ; \; i \in supp(\mathbf{v}))$. Then, the vector $\mathbf{t} = \mathbf{u} - \lambda\mathbf{v}$ satisfies that $supp(\mathbf{t}) \subset supp(\mathbf{u})$. By multiplying a suitable positive integer z, we have an integer vector $\mathbf{t}' = z \cdot \mathbf{t} \in \mathbb{Z}^n$. Then, it follows that $\mathbf{x}^{\mathbf{t}'^+} - \mathbf{x}^{\mathbf{t}'^-}$ belongs to $I_{\mathcal{A}}$. Since the cardinality of the support of $\mathbf{t}'$ is at most $r-1$, we have $\mathbf{t}' = \mathbf{0}$. Hence, $\mathbf{u} = \lambda\mathbf{v}$ and it follows that λ is a positive integer and that $f = (\mathbf{x}^{\mathbf{v}^+})^{\lambda} - (\mathbf{x}^{\mathbf{v}^-})^{\lambda}$. Thus, g satisfies the above condition.

We now use induction on r. Let $h = \mathbf{x}^{\mathbf{s}^+} - \mathbf{x}^{\mathbf{s}^-} \in I_{\mathcal{A}}$ be a circuit satisfying $supp(\mathbf{s}) \subset supp(\mathbf{u})$. By changing the sign of $\mathbf{s}$, we may assume that at least one of u_i/s_i is positive. Let $\lambda = \min(u_i/s_i > 0 \; ; \; i \in supp(\mathbf{s}))$. Then, the vector $\mathbf{t} = \mathbf{u} - \lambda\mathbf{s}$ satisfies that $supp(\mathbf{t}^+) \subset supp(\mathbf{u}^+)$ and $supp(\mathbf{t}^-) \subset supp(\mathbf{u}^-)$. Then, the cardinality of the support of $\mathbf{t}$ is at most $r - 1$. By multiplying a suitable positive integer z, we have an integer vector $\mathbf{t}' = z \cdot \mathbf{t} \in \mathbb{Z}^n$. Then, it follows that $h' = \mathbf{x}^{\mathbf{t}'^+} - \mathbf{x}^{\mathbf{t}'^-}$ belongs to $I_{\mathcal{A}}$. By the hypothesis of induction, there exists a circuit $g = \mathbf{x}^{\mathbf{v}^+} - \mathbf{x}^{\mathbf{v}^-} \in I_{\mathcal{A}}$ such that $supp(\mathbf{v}^+) \subset supp(\mathbf{t}'^+)$, $supp(\mathbf{v}^-) \subset supp(\mathbf{t}'^-)$ and $supp(in_{\prec}(g)) \subset supp(in_{\prec}(h'))$. If $in_{\prec}(h') = \mathbf{x}^{\mathbf{t}'^+}$, then $in_{\prec}(g) = \mathbf{x}^{\mathbf{v}^+}$ and g satisfies the condition above. If $in_{\prec}(h') = \mathbf{x}^{\mathbf{t}'^-}$, then replacing above h by g, we repeat the same argument as above. Since $\mathbf{u} = 1/z\mathbf{t}' + \lambda\mathbf{s}$ and both $1/z$ and λ are positive, it follows that either $in_{\prec}(h') = \mathbf{x}^{\mathbf{t}'^+}$ or $in_{\prec}(h) = \mathbf{x}^{\mathbf{s}^+}$ as desired. Q. E. D.

The following proposition directly follows from Lemma 1.3. Thanks to this proposition, it turns out that, in Proposition 0.1 ([Stu, Theorem 8.3]), it is sufficient to consider only the set of circuits $C_{\mathcal{A}}$ instead of $I_{\mathcal{A}}$.

Proposition 1.4. *Let $C_{\mathcal{A}}$ be the set of circuits of $I_{\mathcal{A}}$ and let $\prec$ be a term order. Then, we have $\sqrt{in_{\prec}(I_{\mathcal{A}})} = \sqrt{in_{\prec}(C_{\mathcal{A}})}$.*

However, note that $in_{\prec}(I_{\mathcal{A}}) \neq in_{\prec}(C_{\mathcal{A}})$ in general.

Example 1.5. Let $\mathcal{A} = \{(2,0,0,0), (1,1,0,0), (1,0,1,0), (0,1,0,1), (0,0,1,1), (0,0,0,2)\} \subset \mathbb{Z}^4$. Then, $C_{\mathcal{A}} = \{x_1x_4^2 - x_2^2x_6, x_1x_5^2 - x_3^2x_6, x_2x_5 - x_3x_4\}$ and $I_{\mathcal{A}} = (x_1x_4^2 - x_2^2x_6, x_1x_5^2 - x_3^2x_6, x_2x_5 - x_3x_4, x_1x_4x_5 - x_2x_3x_6)$. Let $\succ$ be the lexicographic term order induced by $x_1 \succ x_2 \succ \cdots \succ x_6$. Then, we can check that the initial ideal $in_{\succ}(I_{\mathcal{A}}) = (x_1x_4^2, x_1x_5^2, x_2x_5, x_1x_4x_5) \neq (x_1x_4^2, x_1x_5^2, x_2x_5) = in_{\succ}(C_{\mathcal{A}})$ and $\sqrt{in_{\succ}(I_{\mathcal{A}})} = \sqrt{in_{\succ}(C_{\mathcal{A}})} = (x_1x_4, x_1x_5, x_2x_5)$.

In the rest of this section, we discuss basic properties of circuits which is important in the following sections.

Proposition 1.6. *Let $f = \mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-}$ be a binomial in $I_{\mathcal{A}}$. Then, there exists a monomial $M \in K[\mathbf{x}]$ and a positive integer m such that*

$$(\mathbf{x}^{\mathbf{u}^+})^m - (\mathbf{x}^{\mathbf{u}^-})^m = M\left(\prod_{i=1}^{p}(\mathbf{x}^{\mathbf{v}_i{}^+})^{m_i} - \prod_{i=1}^{p}(\mathbf{x}^{\mathbf{v}_i{}^-})^{m_i}\right)$$

where each $m_i \in \mathbb{N}$, each $\mathbf{x}^{\mathbf{v}_i{}^+} - \mathbf{x}^{\mathbf{v}_i{}^-} \in C_{\mathcal{A}}$ and $1 \leq p \leq n-d$.

Proof. Factoring out common variables of $\mathbf{x}^{\mathbf{u}^+}$ and $\mathbf{x}^{\mathbf{u}^-}$, we can find the monomial M. Hence, we may assume that $\mathbf{x}^{\mathbf{u}^+}$ and $\mathbf{x}^{\mathbf{u}^-}$ are relatively prime.

Let ℓ be the minimum number of the cardinality of support of the binomial in $I_{\mathcal{A}}$. Let r be the cardinality of the support of $\mathbf{u}$. Thanks to the proof of Lemma 1.3, if $r = \ell$, then there exists a circuit $\mathbf{x}^{\mathbf{v}^+} - \mathbf{x}^{\mathbf{v}^-} \in C_{\mathcal{A}}$ such that $f = (\mathbf{x}^{\mathbf{v}^+})^{m_1} - (\mathbf{x}^{\mathbf{v}^-})^{m_1}$ where $m_1 \in \mathbb{N}$.

We now use induction on r. By virtue of Lemma 1.3, there exists a circuit $\mathbf{x}^{\mathbf{v}^+} - \mathbf{x}^{\mathbf{v}^-}$ such that $supp(\mathbf{v}^+) \subset supp(\mathbf{u}^+)$ and $supp(\mathbf{v}^-) \subset supp(\mathbf{u}^-)$. Let $\lambda = \min(u_i/v_i > 0\ ;\ i \in supp(\mathbf{v}))$. Then, the vector $\mathbf{t} = \mathbf{u} - \lambda\mathbf{v}$ satisfies that $supp(\mathbf{t}^+) \subset$

$supp(\mathbf{u}^+)$ and $supp(\mathbf{t}^-) \subset supp(\mathbf{u}^-)$. By multiplying a suitable positive integer z, we have an integer vector $\mathbf{t}' = z \cdot \mathbf{t} \in \mathbb{Z}^n$. Then, it follows that $\mathbf{x}^{\mathbf{t}'^+} - \mathbf{x}^{\mathbf{t}'^-}$ belongs to $I_{\mathcal{A}}$. Since $supp(\mathbf{t}') \neq supp(\mathbf{u})$, the hypothesis of induction enables us to show that there exists a positive integer m such that

$$(\mathbf{x}^{\mathbf{t}'^+})^m - (\mathbf{x}^{\mathbf{t}'^-})^m = \prod_{i=1}^{p} (\mathbf{x}^{\mathbf{v}_i^+})^{m_i} - \prod_{i=1}^{p} (\mathbf{x}^{\mathbf{v}_i^-})^{m_i}$$

where each $m_i \in \mathbb{N}$, each $\mathbf{x}^{\mathbf{v}_i^+} - \mathbf{x}^{\mathbf{v}_i^-} \in C_{\mathcal{A}}$. Since $mz\mathbf{u} = m\mathbf{t}' + mz\lambda\mathbf{v}$, we have

$$(\mathbf{x}^{\mathbf{u}^+})^{mz} - (\mathbf{x}^{\mathbf{u}^-})^{mz} = (\mathbf{x}^{\mathbf{v}^+})^{mz\lambda} \cdot \prod_{i=1}^{p} (\mathbf{x}^{\mathbf{v}_i^+})^{m_i} - (\mathbf{x}^{\mathbf{v}^-})^{mz\lambda} \cdot \prod_{i=1}^{p} (\mathbf{x}^{\mathbf{v}_i^-})^{m_i}.$$

Finally, we must show that only $p \leq n - d$ circuits are needed. Suppose that $p > n - d$. Since each vector $\mathbf{v}_i$ belongs to the $(n-d)$-dimensional vector space $\{\mathbf{u} \in \mathbf{Q}^n \ ; \ \mathcal{A}\mathbf{u} = \mathbf{0}\}$, there exists a linear dependence $\sum_{j \in J} n_j v_j = 0$ where $0 < n_i \in \mathbb{N}$ and $\emptyset \neq J \subset \{1, 2, \ldots, p\}$. Let $N = \min(m_j/n_j > 0 \ ; \ j \in J)$. We define the vector $M = (m_1, m_2, \ldots, m_p) - N(n_1, n_2, \ldots, n_p) \in \mathbf{Q}^p$ where $n_j = 0$ if $j \notin J$. By multiplying a suitable positive integer q, we have an integer vector $M' = q \cdot M = (m_1', m_2', \ldots, m_p') \in \mathbb{Z}^p$. Then, the cardinality of $supp(M')$ is at most $p - 1$ and

$$(\mathbf{x}^{\mathbf{u}^+})^{mq} - (\mathbf{x}^{\mathbf{u}^-})^{mq} = \prod_{i \in supp(M')} (\mathbf{x}^{\mathbf{v}_i^+})^{m_i'} - \prod_{i \in supp(M')} (\mathbf{x}^{\mathbf{v}_i^-})^{m_i'}$$

as desired. Q. E. D.

$\mathcal{A}$ is called *unimodular* if all triangulations of $\mathcal{P}_{\mathcal{A}}$ are unimodular. It is known that $\mathcal{A}$ is unimodular if and only if both terms of any circuits of $I_{\mathcal{A}}$ are squarefree. If $\mathcal{A}$ is unimodular, then we immediately have the following:

Corollary 1.7. *Let $f = \mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-} \in I_{\mathcal{A}}$ where $\mathbf{x}^{\mathbf{u}^+}$ and $\mathbf{x}^{\mathbf{u}^-}$ are relatively prime. If $\mathcal{A}$ is unimodular, then we have*

$$\mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-} = \prod_{i=1}^{p} (\mathbf{x}^{\mathbf{v}_i^+})^{m_i} - \prod_{i=1}^{p} (\mathbf{x}^{\mathbf{v}_i^-})^{m_i}$$

where each $m_i \in \mathbb{N}$, each $\mathbf{x}^{\mathbf{v}_i^+} - \mathbf{x}^{\mathbf{v}_i^-} \in C_{\mathcal{A}}$ and $1 \leq p \leq n - d$.

Proof. Since both terms of every circuit of $I_{\mathcal{A}}$ are squarefree, in the proof of Proposition 1.6, $\lambda = \min(u_i/v_i > 0 \ ; \ i \in supp(\mathbf{v})) = \min(|u_i| \ ; u_i/v_i > 0, \ i \in supp(\mathbf{v})) \in \mathbb{N}$ since each $v_i = \pm 1$. Hence, we have $z = 1$. Moreover, by the hypothesis of induction, we may assume that $m = 1$. Q. E. D.

Lemma 1.8. *Let $I, J \subset \{1, 2, \ldots, n\}$ with $I \neq J$ and suppose that the equation*

$$\sum_{i \in I} a_i \mathbf{a}_i = \sum_{j \in J} b_j \mathbf{a}_j,$$

holds, where $0 < a_i, b_j \in \mathbf{Q}$. *Then, there exists a circuit* $\mathbf{x}^{\mathbf{v}^+} - \mathbf{x}^{\mathbf{v}^-} \in C_{\mathcal{A}}$ *such that* $supp(\mathbf{x}^{\mathbf{v}^+}) \subset I$ *and* $supp(\mathbf{x}^{\mathbf{v}^-}) \subset J$.

Proof. By multiplying a suitable integer to the equation above, we have

$$\sum_{i \in I} a'_i \mathbf{a}_i = \sum_{j \in J} b'_j \mathbf{a}_j$$

where $0 < a'_i, b'_j \in \mathbb{Z}$ and $\sum_{i \in I} a'_i = \sum_{j \in J} b'_j$. Hence, the homogeneous binomial

$$\prod_{i \in I} x_i^{a'_i} - \prod_{j \in J} x_j^{b'_j} \neq 0$$

belongs to $I_{\mathcal{A}}$. Thanks to Lemma 1.3, there exists a circuit $\mathbf{x}^{v^+} - \mathbf{x}^{v^-} \in C_{\mathcal{A}}$ such that $supp(v^+) \subset I$ and $supp(v^-) \subset J$. Q. E. D.

2 Triangulations and markings

In this section, we study the relation between triangulations and markings on $C_{\mathcal{A}}$. If σ is a convex polytope, then let $V(\sigma)$ denote the set of all vertices of σ. We often identify $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ with $\{1, \ldots, n\}$. Thanks to Proposition 1.4, the following theorem generalizes [Stu, Theorem 8.3].

Theorem 2.1. *Let* Δ *be a triangulation of* $\mathcal{P}_{\mathcal{A}}$. *Then, there exists a marking* $in(\cdot)$ *on* $C_{\mathcal{A}}$ *such that* $I_\Delta = \sqrt{in(C_{\mathcal{A}})}$. *In particular,* Δ *is regular if and only if there exists a coherent marking* $in(\cdot)$ *on* $C_{\mathcal{A}}$ *such that* $I_\Delta = \sqrt{in(C_{\mathcal{A}})}$.

Proof. Let a binomial $f = \mathbf{x}^{v^+} - \mathbf{x}^{v^-}$ belong to $I_{\mathcal{A}}$. Then, we have

$$\sum_{i \in supp(v^+)} v_i \mathbf{a}_i = \sum_{j \in supp(v^-)} v_j \mathbf{a}_j$$

where $0 < v_i \in \mathbb{Z}$. Moreover, since $I_{\mathcal{A}}$ is homogeneous in the usual grading, we have $\sum_{i \in supp(v^+)} v_i = \sum_{j \in supp(v^-)} v_j$. Hence, two polytopes $supp(v^+)$ and $supp(v^-)$ intersect in their interior. Since Δ is a triangulation, either $supp(v^+)$ or $supp(v^-)$ is a nonface of Δ, i.e., either $\mathbf{x}^{v^+}$ or $\mathbf{x}^{v^-}$ belongs to I_Δ.

We consider a marking $in(\cdot)$ such that $in(f) \in I_\Delta$ for each $f \in C_{\mathcal{A}}$. Suppose that σ is a minimal nonface of Δ. Now, we choose a point $\sum_{\mathbf{a}_i \in V(\sigma)} a_i \mathbf{a}_i \in \sigma \subset \mathcal{P}_{\mathcal{A}}$ where $0 < a_i \in \mathbf{Q}$ and $\sum_{\mathbf{a}_i \in V(\sigma)} a_i = 1$. Since Δ is a triangulation, there exists a simplex $\sigma' \in \Delta$ such that

$$\sum_{\mathbf{a}_i \in V(\sigma)} a_i \mathbf{a}_i = \sum_{\mathbf{a}_j \in V(\sigma')} b_j \mathbf{a}_j \tag{1}$$

where $0 < b_j \in \mathbf{Q}$ and $\sum_{\mathbf{a}_j \in \sigma'} b_j = 1$. Thanks to Lemma 1.8, there exists a circuit $f = \mathbf{x}^{v^+} - \mathbf{x}^{v^-} \in C_{\mathcal{A}}$ such that $supp(v^+) \subset V(\sigma)$ and $supp(v^-) \subset V(\sigma')$. Since $\sigma' \in \Delta$, we have $\mathbf{x}^{v^-} \notin I_\Delta$ and $\mathbf{x}^{v^+} \in I_\Delta$. Hence, $supp(v^+) = V(\sigma)$ and $in(f) = \mathbf{x}^{v^+}$. Thus, we have $I_\Delta = \sqrt{in(C_{\mathcal{A}})}$. Moreover, by virtue of Proposition 1.4 and [Stu, Theorem 8.3], Δ is regular if and only if there exists a coherent marking $in(\cdot)$ on $C_{\mathcal{A}}$ such that $I_\Delta = \sqrt{in(I_{\mathcal{A}})} = \sqrt{in(C_{\mathcal{A}})}$. Q. E. D.

The converse of Theorem 2.1 is false in general, i.e., there is a marking $in(\cdot)$ such that $\sqrt{in(C_{\mathcal{A}})}$ is not the Stanley–Reisner ideal of any triangulation of $\mathcal{P}_{\mathcal{A}}$.

Example 2.2. Let $\mathcal{A} = \{(1,0,1,0,0), (1,0,0,1,0), (1,0,0,0,1), (0,1,1,0,0), (0,1,0,1,0), (0,1,0,0,1)\} \subset \mathbb{Z}^5$. Then, $C_{\mathcal{A}} = \{f_1 = x_1x_5 - x_2x_4, f_2 = x_2x_6 - x_3x_5, f_3 = x_3x_4 - x_1x_6\}$. Now, we consider a noncoherent marking $in(\cdot)$ defined by $in(f_1) = x_1x_5, in(f_2) = x_2x_6$ and $in(f_3) = x_3x_4$. Then, $\sqrt{in(C_{\mathcal{A}})} = (x_1x_5, x_2x_6, x_3x_4)$. Suppose that there exists a triangulation Δ of $\mathcal{P}_{\mathcal{A}}$ with $I_\Delta = \sqrt{in(C_{\mathcal{A}})}$. Then, the set of vertices of maximal simplices in Δ is $\{\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}, \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4\}, \{\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_6\}, \{\mathbf{a}_1, \mathbf{a}_4, \mathbf{a}_6\}, \{\mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_5\}, \{\mathbf{a}_2, \mathbf{a}_4, \mathbf{a}_5\}, \{\mathbf{a}_3, \mathbf{a}_5, \mathbf{a}_6\}, \{\mathbf{a}_4, \mathbf{a}_5, \mathbf{a}_6\}\}$. Since $\dim \mathcal{P}_{\mathcal{A}} = 3$, this is a contradiction. Thus, there exists no triangulation Δ of $\mathcal{P}_{\mathcal{A}}$ with $I_\Delta = \sqrt{in(C_{\mathcal{A}})}$.

However, note that, *geometrically*, Δ in Example 2.2 is a simplicial complex on the vertex set $\mathcal{A}$. Let $\Delta_{in} = \left\{\mathcal{P}_{\mathcal{B}} \subset \mathbb{R}^d \ ; \ \mathcal{B} \subset \mathcal{A}, \prod_{\mathbf{a}_i \in \mathcal{B}} x_i \notin \sqrt{in(C_{\mathcal{A}})} \ \right\}$.

Proposition 2.3. *Let $in(\cdot)$ be a marking on $C_{\mathcal{A}}$. Then, Δ_{in} is a simplicial complex on the vertex set $\mathcal{A}$ such that $I_{\Delta_{in}} = \sqrt{in(C_{\mathcal{A}})}$.*

Proof. If $\sigma \in \Delta_{in}$ is not a subsimplex of $\mathcal{P}_{\mathcal{A}}$, then there exists an affine dependence on the vertices of σ, i.e., there exist two disjoint subsets J_1 and J_2 of $V(\sigma)$ such that

$$\sum_{\mathbf{a}_i \in J_1} a_i \mathbf{a}_i = \sum_{\mathbf{a}_j \in J_2} b_j \mathbf{a}_j$$

where $0 < a_i, b_j \in \mathbf{Q}$ and $\sum_{\mathbf{a}_i \in J_1} a_i = \sum_{\mathbf{a}_j \in J_2} b_j = 1$. Thanks to Lemma 1.8, there exists a circuit $f = \mathbf{x}^{v^+} - \mathbf{x}^{v^-} \in C_{\mathcal{A}}$ such that $supp(v^+) \subset J_1$ and $supp(v^-) \subset J_2$. Then, neither $\mathbf{x}^{v^+}$ nor $\mathbf{x}^{v^-}$ belongs to $\sqrt{in(C_{\mathcal{A}})}$. This is a contradiction. Hence, σ is a subsimplex of $\mathcal{P}_{\mathcal{A}}$.

Let $\sigma_1, \sigma_2 \in \Delta_{in}$. We choose a point $\alpha \in \sigma_1 \cap \sigma_2$. Then, we have

$$\alpha = \sum_{\mathbf{a}_i \in V(\sigma_1)} a_i \mathbf{a}_i = \sum_{\mathbf{a}_j \in V(\sigma_2)} b_j \mathbf{a}_j,$$

where $0 \leq a_i, b_j \in \mathbf{Q}$ and $\sum_{\mathbf{a}_i \in V(\sigma_1)} a_i = \sum_{\mathbf{a}_j \in V(\sigma_2)} b_j = 1$. By multiplying a suitable integer to the equation above, we have

$$\sum_{\mathbf{a}_i \in V(\sigma_1)} a'_i \mathbf{a}_i = \sum_{\mathbf{a}_j \in V(\sigma_2)} b'_j \mathbf{a}_j$$

where $0 \le a'_i, b'_j \in \mathbb{Z}$ and $\sum_{\mathbf{a}_i \in V(\sigma_1)} a'_i = \sum_{\mathbf{a}_j \in V(\sigma_2)} b'_j$. If

$$g = \prod_{\mathbf{a}_i \in V(\sigma_1)} x_i^{a'_i} - \prod_{\mathbf{a}_j \in V(\sigma_2)} x_j^{b'_j} \neq 0,$$

then the homogeneous binomial g belongs to $I_{\mathcal{A}}$. Thanks to Lemma 1.3, there exists a circuit $f = \mathbf{x}^{v^+} - \mathbf{x}^{v^-} \in C_{\mathcal{A}}$ such that $supp(v^+) \subset V(\sigma_1)$ and $supp(v^-) \subset V(\sigma_2)$. Then, neither $\mathbf{x}^{v^+}$ nor $\mathbf{x}^{v^-}$ belongs to $\sqrt{in(C_{\mathcal{A}})}$. This is a contradiction. Hence, we have $g = 0$. Thus, α has a representation

$$\alpha = \sum_{\mathbf{a}_i \in V(\sigma_1) \cap V(\sigma_2)} a_i \mathbf{a}_i.$$

Thus, Δ_{in} is a simplicial complex on the vertex set $\mathcal{A}$. Q. E. D.

If Δ_{in} is a triangulation of $\mathcal{P}_{\mathcal{A}}$, then we call $in(\cdot)$ a *geometric marking* on $C_{\mathcal{A}}$. For a marking $in(\cdot)$ on $C_{\mathcal{A}}$, we define the subsets $\mathcal{G}_{in}^{(1)}$ and $\mathcal{G}_{in}^{(2)}$ of $C_{\mathcal{A}}$ as follows:

$$\begin{aligned} \mathcal{G}_{in}^{(1)} &= \{ f \in C_{\mathcal{A}} \ ; \ f - in(f) \notin \sqrt{in(C_{\mathcal{A}})} \ \} \\ \mathcal{G}_{in}^{(2)} &= \{ f \in \mathcal{G}_{in}^{(1)} \ ; \ \text{there exists no } g \in C_{\mathcal{A}} \text{ such that } supp(in(g)) \subsetneq supp(in(f)) \}. \end{aligned}$$

Then, $f \in \mathcal{G}_{in}^{(1)}$ (resp. $\mathcal{G}_{in}^{(2)}$) satisfies that $supp(in(f))$ is a nonface (resp. minimal nonface) of Δ_{in} and $supp(f - in(f))$ is a face of Δ_{in}. Note that if $\mathcal{A}$ is unimodular and $in(\cdot)$ is coherent, then $\mathcal{G}_{in}^{(2)}$ coincides with the reduced Gröbner basis of $I_{\mathcal{A}}$ with respect to the term order $in(\cdot)$.

Theorem 2.4. *Suppose that $in(\cdot)$ is a geometric marking on $C_{\mathcal{A}}$. Then, we have $\sqrt{in(\mathcal{G}_{in}^{(2)})} = \sqrt{in(C_{\mathcal{A}})}$.*

Proof. In the proof of Theorem 2.1, we have shown that if σ is a minimal nonface of Δ_{in}, then there exists a circuit $f = \mathbf{x}^{v^+} - \mathbf{x}^{v^-} \in C_{\mathcal{A}}$ such that $supp(v^+) = V(\sigma)$ and $\mathbf{x}^{v^-} \notin \sqrt{in(C_{\mathcal{A}})}$. Since $f \in \mathcal{G}_{in}^{(2)}$, this completes the proof. Q. E. D.

Corollary 2.5. *Let $in(\cdot)$ and $in'(\cdot)$ be geometric markings on $C_{\mathcal{A}}$. Then, $\Delta_{in} = \Delta_{in'}$ if and only if $\mathcal{G}_{in}^{(2)} = \mathcal{G}_{in'}^{(2)}$ and $in(g) = in'(g)$ for all $g \in \mathcal{G}_{in}^{(2)}$.*

Proof. Since $\mathcal{G}_{in}^{(2)}$ is uniquely determined by Δ_{in}, "only if" part holds. On the other hand, Theorem 2.4 enables us to show "if" part. Q. E. D.

Now, we study a criterion for a marking on $C_{\mathcal{A}}$ to be a geometric marking. Note that, in the conditions (ii) and (iii) below, m' is unique for each $\{m, p\}$, $\{m, p'\}$.

Theorem 2.6. *For a marking $in(\cdot)$ on $C_{\mathcal{A}}$, the following conditions are equivalent:*

(i) *$in(\cdot)$ is a geometric marking;*

(ii) *there exists a positive integer* p *and a monomial* $m' \notin \sqrt{in(C_{\mathcal{A}})}$ *such that* $m^p - m' \in I_{\mathcal{A}}$ *for an arbitrary monomial* $m \in in(C_{\mathcal{A}})$;

(iii) *there exists a positive integer* p' *and a sequence of reductions from* $m^{p'}$ *to* $m' \notin \sqrt{in(C_{\mathcal{A}})}$ *modulo* $\mathcal{G}_{in}^{(1)}$ *for an arbitrary monomial* $m \in in(C_{\mathcal{A}})$.

Moreover, if $\mathcal{A}$ *is unimodular, then we have* $p = p' = 1$.

Proof. First, by virtue of Proposition 1.6 and Corollary 1.7, we have (ii) $\Leftrightarrow$ (iii).

Second, we show that (i) $\Rightarrow$ (ii). Suppose that $in(\cdot)$ on $C_{\mathcal{A}}$ is a geometric marking. Let $m = \prod_{i=1}^n x_i^{a_i} \in in(C_{\mathcal{A}})$, $s = \sum_{i=1}^n a_i$ and $a_i' = a_i/s$. Then, we have $\alpha = \sum_{i \in supp(m)} a_i' \mathbf{a}_i \in \mathcal{P}_{\mathcal{A}}$ since $0 < a_i' \in \mathbf{Q}$ and $\sum_{i \in supp(m)} a_i' = 1$. Since Δ_{in} is a triangulation, there exists a unique simplex $\sigma \in \Delta_{in}$ such that $\alpha = \sum_{i \in supp(m)} a_i' \mathbf{a}_i = \sum_{\mathbf{a}_j \in V(\sigma)} b_j \mathbf{a}_j$, where $0 < b_j \in \mathbf{Q}$ and $\sum_{\mathbf{a}_j \in V(\sigma)} b_j = 1$. By the similar argument as in the proof of Lemma 1.8, there exists a positive integer p such that $f = m^p - \prod_{\mathbf{a}_j \in V(\sigma)} x_j^{b_j s p} \in I_{\mathcal{A}}$. Since $\sigma \in \Delta$, we have $\prod_{\mathbf{a}_j \in V(\sigma)} x_j^{b_j s p} \notin \sqrt{in(C_{\mathcal{A}})}$.

Suppose that $\mathcal{A}$ is unimodular. Since $s \cdot \alpha \in \mathbb{Z}\mathcal{A}$ and σ is a face of a simplex of normalized volume 1, we have $b_j s \in \mathbb{N}$ for all j with $\mathbf{a}_j \in V(\sigma)$. Hence, we have $f' = m - \prod_{\mathbf{a}_j \in V(\sigma)} x_j^{b_j s} \in I_{\mathcal{A}}$ and $\prod_{\mathbf{a}_j \in V(\sigma)} x_j^{b_j s} \notin \sqrt{in(C_{\mathcal{A}})}$.

Finally, we show that (ii) $\Rightarrow$ (i). Choose an arbitrary point $\alpha = \sum_{i=1}^n a_i \mathbf{a}_i \in \mathcal{P}_{\mathcal{A}}$ where $0 \le a_i \in \mathbf{Q}$ and $\sum_{i=1}^n a_i = 1$. By multiplying a suitable integer z, we have $z \cdot \alpha = \sum_{i=1}^n a_i' \mathbf{a}_i$ where $0 \le a_i' \in \mathbb{Z}$. Now, we consider the monomial $m = \prod_{i=1}^n x_i^{a_i'}$. By the hypothesis, there exists a positive integer p and a monomial $m' = \prod_{j \in J} x_j^{b_j} \notin \sqrt{in(C_{\mathcal{A}})}$ such that $m^p - m' \in I_{\mathcal{A}}$. Hence, we have

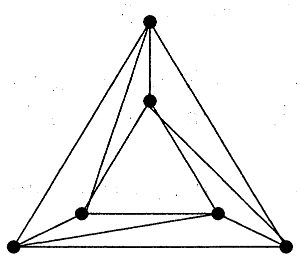
$$\alpha = \sum_{i=1}^n a_i \mathbf{a}_i = \sum_{j \in J} \left(\frac{b_j}{z\, p} \right) \mathbf{a}_j.$$

Since $m' \notin \sqrt{in(C_{\mathcal{A}})}$, we have $J \in \Delta_{in}$. Thus, we have $\mathcal{P}_{\mathcal{A}} = \bigcup_{\sigma \in \Delta_{in}} \sigma$. Thanks to Proposition 2.3, Δ_{in} is a triangulation of $\mathcal{P}_{\mathcal{A}}$ as desired. Q. E. D.

Example 2.7. Let $\mathcal{A} = \{(6,0,0), (0,6,0), (0,0,6), (4,1,1), (1,4,1), (1,1,4)\} \subset \mathbb{Z}^3$. Then, $\mathcal{P}_{\mathcal{A}}$ is a planar triangle and $C_{\mathcal{A}}$ consists of the following 15 circuits:

$$\begin{aligned} C_{\mathcal{A}} = \{ & x_2x_4^2 - x_1x_5^2, x_3x_5^2 - x_2x_6^2, x_1x_6^2 - x_3x_4^2, x_1^4x_2x_3 - x_4^6, x_1x_2^4x_3 - x_5^6, \\ & x_1x_2x_3^4 - x_6^6, x_1^3x_5x_6 - x_4^5, x_2^3x_4x_6 - x_5^5, x_3^3x_4x_5 - x_6^5, x_1^5x_2x_6^2 - x_4^8, \\ & x_1x_2^5x_6^2 - x_5^8, x_2^5x_3x_4^2 - x_5^8, x_2x_3^5x_4^2 - x_6^8, x_1^5x_3x_5^2 - x_4^8, x_1x_3^5x_5^2 - x_6^8 \}. \end{aligned}$$

Now, we consider the following triangulation Δ of $\mathcal{P}_{\mathcal{A}}$:



If we define the marking $in(\cdot)$ as $in(f)$ is the first term in the above expression for each $f \in C_{\mathcal{A}}$, then we have $I_\Delta = \sqrt{in(C_{\mathcal{A}})} = (x_2x_4, x_3x_5, x_1x_6, x_1x_2x_3)$. In this case, $\mathcal{G}_{in}^{(1)} = C_{\mathcal{A}}$ and $\mathcal{G}_{in}^{(2)} = \{x_2x_4^2 - x_1x_5^2, x_3x_5^2 - x_2x_6^2, x_1x_6^2 - x_3x_4^2, x_1^4x_2x_3 - x_4^6, x_1x_2^4x_3 - x_5^6, x_1x_2x_3^4 - x_6^6\}$. The triangulation Δ is nonregular because $in(\cdot)$ on $\{x_2x_4^2 - x_1x_5^2, x_3x_5^2 - x_2x_6^2, x_1x_6^2 - x_3x_4^2\} \subset \mathcal{G}_{in}^{(2)}$ is not coherent. Note the following two sequences of reductions:

$$x_2x_3x_4^2 \xrightarrow[x_2x_4^2-x_1x_5^2]{} x_1x_3x_5^2 \xrightarrow[x_3x_5^2-x_2x_6^2]{} x_1x_2x_6^2 \xrightarrow[x_1x_6^2-x_3x_4^2]{} x_2x_3x_4^2$$

$$(x_2x_3x_4^2)^{15} \xrightarrow[x_2^5x_3x_4^2-x_5^8]{} x_3^{12}x_4^{24}x_5^{24} \xrightarrow[x_3^3x_4x_5-x_6^5]{} x_4^{20}x_5^{20}x_6^{20} \notin \sqrt{in(C_{\mathcal{A}})}.$$

First sequence of reductions means that $in(\cdot)$ on $\{x_2x_4^2 - x_1x_5^2, x_3x_5^2 - x_2x_6^2, x_1x_6^2 - x_3x_4^2\} \subset \mathcal{G}_{in}^{(2)}$ is not coherent. Second sequence of reductions illustrates Theorem 2.6.

3 Flips

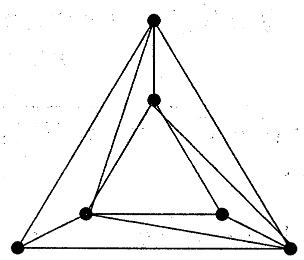
Let Δ be a triangulation of $\mathcal{P}_{\mathcal{A}}$ and let $f = \mathbf{x}^{u^+} - \mathbf{x}^{u^-} \in C_{\mathcal{A}}$ be a circuit. We say that Δ is *supported on f* if the following two conditions are satisfied:

(i) $\prod_{i \in supp(f)\setminus\{j\}} x_i \notin I_\Delta$ for all $j \in supp(\mathbf{x}^{u^+})$;

(ii) For all monomials $m \in K[\mathbf{x}]$ such that $supp(m) \cap supp(\mathbf{x}^{u^+}) = \emptyset$ and for all $j_1, j_2 \in supp(\mathbf{x}^{u^+})$, we have $m \cdot \prod_{i \in supp(f)\setminus\{j_1\}} x_i$ belongs to I_Δ if and only if $m \cdot \prod_{i \in supp(f)\setminus\{j_2\}} x_i$ belongs to I_Δ.

If a triangulation Δ of $\mathcal{P}_{\mathcal{A}}$ is supported on a circuit $f \in C_{\mathcal{A}}$, then we can construct a new triangulation by taking away all the simplices of the form $supp(m) \cup supp(f) \setminus \{i\}$ where $i \in supp(\mathbf{x}^{u^+})$ and $m \in K[\mathbf{x}]$ is a monomial with $supp(m) \cap supp(f) = \emptyset$ and adding the simplices of the form $supp(m) \cup supp(f) \setminus \{j\}$ where $j \in supp(\mathbf{x}^{u^-})$ and the same m. We call this operation a *flip* along f. See [G–K–Z] for the details.

Example 3.1. We continue the argument in Example 2.7. The triangulation Δ is supported on circuits $\{x_2x_4^2 - x_1x_5^2, x_3x_5^2 - x_2x_6^2, x_1x_6^2 - x_3x_4^2\}$. For example, the triangulation Δ' obtained by a flip from Δ along the circuit $x_3x_5^2 - x_2x_6^2$ is as follows:



Then, Δ' is a regular triangulation of $\mathcal{P}_{\mathcal{A}}$.

First, we represent a flip of a triangulation Δ of $\mathcal{P}_{\mathcal{A}}$ as a operation for a geometric marking $in(\cdot)$ such that $\Delta = \Delta_{in}$. Suppose that a triangulation Δ_{in} is supported on a circuit $f \in C_{\mathcal{A}}$ and Δ' is obtained by a flip from Δ_{in} along f. Now, we define a marking $in'(\cdot)$ on $C_{\mathcal{A}}$ by

$$in'(g) = \begin{cases} g - in(g) & g = f \\ g - in(g) & g \notin \mathcal{G}_{in}^{(1)} \text{ and } supp(in(f)) \subset supp(in(g)) \\ in(g) & \text{otherwise.} \end{cases} \tag{3}$$

Theorem 3.2. *Work with the same situation as above. Then, $\Delta' = \Delta_{in'}$.*

Proof. For a monomial m, if $m \in I_{\Delta}$ and if $m \notin I_{\Delta'}$, then $supp(in(f)) \subset supp(m)$. Similarly, if $m \notin I_{\Delta}$ and if $m \in I_{\Delta'}$, then $supp(f - in(f)) \subset supp(m)$.

Suppose that there exists a circuit $g \in C_{\mathcal{A}}$ such that $in'(g) \notin I_{\Delta'}$. It then follows that $g - in'(g) \in I_{\Delta'}$. By the definition of flips, we have $in'(f) = f - in(f) \in I_{\Delta'}$. Hence, we have $g \neq f$.

Suppose that $g \in \mathcal{G}_{in}^{(1)}$. By the definition (3), $in'(g) = in(g) \in I_{\Delta}$. Hence, we have $supp(in(f)) \subset supp(in'(g))$. Moreover, since $g - in'(g) \notin I_{\Delta}$, we have $supp(f - in(f)) \subset supp(g - in'(g))$. This contradicts that g is a circuit.

Suppose that $g \notin \mathcal{G}_{in}^{(1)}$, i.e., both $in(g)$ and $g - in(g)$ belong to I_{Δ}. Since $in'(g) \notin I_{\Delta'}$, we have $supp(in(f)) \subset supp(in'(g))$. If $in'(g) = in(g)$, then $supp(in(f)) \subset supp(in'(g)) = supp(in(g))$. This contradicts the definition (3). If $in'(g) = g - in(g)$, then we have $supp(in(f)) \subset supp(in(g))$ by the definition (3). Since $supp(in(f)) \subset supp(g - in(g))$ and $supp(in(f)) \subset supp(in(g))$, this contradicts that g is a circuit.

Hence, we have $\sqrt{in'(C_{\mathcal{A}})} \subset I_{\Delta'}$. Thus, $\Delta' \subset \Delta_{in'}$. Thanks to Proposition 2.3 and since Δ' is a triangulation, $\Delta' = \Delta_{in'}$ as desired. Q. E. D.

Theorem 3.3. *Suppose that a triangulation Δ' is obtained by a flip from a triangulation Δ along the circuit $f \in C_{\mathcal{A}}$. Then, there exist markings $in(\cdot)$ and $in'(\cdot)$ such that $\Delta = \Delta_{in}$, $\Delta' = \Delta_{in'}$, $in(f) = f - in'(f)$ and $in(g) = in'(g)$ for all $g \in C_{\mathcal{A}} \setminus \{f\}$.*

Proof. Let $in(\cdot)$ and $in'(\cdot)$ be markings with $\Delta = \Delta_{in}$ and $\Delta' = \Delta_{in'}$, i.e., work with the same situation in (3). Let $in^*(\cdot)$ be a marking defined by

$$in^*(g) = \begin{cases} g - in(g) & g \notin \mathcal{G}_{in}^{(1)} \text{ and } supp(in(f)) \subset supp(in(g)) \\ in(g) & \text{otherwise.} \end{cases}$$

Thanks to Theorem 2.4, we have $\sqrt{in(C_{\mathcal{A}})} = \sqrt{in^*(C_{\mathcal{A}})}$. Hence, we have $\Delta_{in} = \Delta_{in^*}$ and we can rewrite (3) as follows:

$$in'(g) = \begin{cases} g - in^*(g) & g = f \\ in^*(g) & \text{otherwise.} \end{cases}$$

Thus, we have a desired conclusion by Theorem 3.2. Q. E. D.

Now, we want to know which circuits support a triangulation. The following theorem and corollary are related with the S-polynomial $S(f,g)$ of the binomials $f = \mathbf{x}^{u^+} - \mathbf{x}^{u^-} \in I_{\mathcal{A}}$ and $g = \mathbf{x}^{v^+} - \mathbf{x}^{v^-} \in I_{\mathcal{A}}$ with $in(f) = \mathbf{x}^{u^+}$ and $in(g) = \mathbf{x}^{v^+}$:

$$S(f,g) = \frac{LCM(in(f), in(g))}{in(f)} \cdot \mathbf{x}^{u^-} - \frac{LCM(in(f), in(g))}{in(g)} \cdot \mathbf{x}^{v^-}.$$

Theorem 3.4. *Let $in(\cdot)$ be a geometric marking and let $f = \mathbf{x}^{u^+} - \mathbf{x}^{u^-} \in C_{\mathcal{A}}$ with $in(f) = \mathbf{x}^{u^+}$. If $p = max(\ |u_i| \in \mathbb{Z}\ ;\ \mathbf{x}^{u^+} - \mathbf{x}^{u^-} \in C_{\mathcal{A}}\ )$, then a triangulation Δ_{in} of $\mathcal{P}_{\mathcal{A}}$ is supported on f if and only if $f \in \mathcal{G}_{in}^{(2)}$ and*

$$\frac{LCM(in(f)^p, in(g))}{in(f)^p} \cdot \mathbf{x}^{u^-} \in \sqrt{in(C_{\mathcal{A}})}$$

for all $g \in \mathcal{G}_{in}^{(2)} \setminus \{f\}$.

Proof. [**only if**] Suppose that Δ_{in} is supported on a circuit $f = \mathbf{x}^{u^+} - \mathbf{x}^{u^-} \in C_{\mathcal{A}}$. Then, $\prod_{i \in supp(f)\setminus\{j\}} x_i \notin \sqrt{in(C_{\mathcal{A}})}$ for all $j \in supp(\mathbf{x}^{u^+})$. Hence, in particular, neither $\mathbf{x}^{u^-}$ nor $\prod_{i \in supp(\mathbf{x}^{u^+})\setminus\{j\}} x_i$ belongs to $\sqrt{in(C_{\mathcal{A}})}$. Thus, $f \in \mathcal{G}_{in}^{(2)}$.

Suppose that

$$\frac{LCM(in(f)^p, in(g))}{in(f)^p} \cdot \mathbf{x}^{u^-} \notin \sqrt{in(C_{\mathcal{A}})}$$

for a circuit $g = \mathbf{x}^{v^+} - \mathbf{x}^{v^-} \in \mathcal{G}_{in}^{(2)} \setminus \{f\}$ with $in(g) = \mathbf{x}^{v^+}$. Then, $supp(\mathbf{x}^{u^+}) \cap supp(\mathbf{x}^{v^+}) \neq \emptyset$. Suppose that $supp(\mathbf{x}^{u^+}) = supp(\mathbf{x}^{v^+})$. Let $v_r/u_r = \min(v_i/u_i > 0\ ;\ i \in supp(\mathbf{x}^{u^+}))$ where $r \in supp(\mathbf{x}^{u^+})$. It then follows that $(\mathbf{x}^{u^+})^{v_r}$ divides $(\mathbf{x}^{v^+})^{u_r}$. Hence, we have

$$0 \neq \frac{(\mathbf{x}^{v^+})^{u_r}}{(\mathbf{x}^{u^+})^{v_r}} \cdot (\mathbf{x}^{u^-})^{v_r} - (\mathbf{x}^{v^-})^{u_r} \in I_{\mathcal{A}}.$$

Since $(\mathbf{x}^{v^-})^{u_r} \notin \sqrt{in(C_{\mathcal{A}})}$, we have $m = \dfrac{(\mathbf{x}^{v^+})^{u_r}}{(\mathbf{x}^{u^+})^{v_r}} \cdot (\mathbf{x}^{u^-})^{v_r} \in \sqrt{in(C_{\mathcal{A}})}$. Since $supp(m) \subset supp(f) \setminus \{r\}$, Δ_{in} is not supported on f.

Suppose that $supp(\mathbf{x}^{u^+}) \neq supp(\mathbf{x}^{v^+})$ and Δ' is obtained by a flip from Δ_{in} along f. Since $g \in \mathcal{G}_{in}^{(2)}$, $supp(\mathbf{x}^{u^+}) \setminus supp(\mathbf{x}^{v^+}) \neq \emptyset$. If $i \in supp(\mathbf{x}^{u^+}) \setminus supp(\mathbf{x}^{v^+})$, then $supp(LCM(in(f), in(g)) \cdot \mathbf{x}^{u^-}) \setminus \{i\}$ is not a face of Δ_{in}. Hence,

$$\frac{LCM(in(f)^p, in(g))}{in(f)^p} \cdot \mathbf{x}^{u^-} \notin I_{\Delta'}.$$

Thus, $\mathbf{x}^{u^-} \notin I_{\Delta'}$ and this contradicts Theorem 3.3.

[if] Suppose that $m = \prod_{i \in supp(f) \setminus \{j\}} x_i \in \sqrt{in(C_{\mathcal{A}})}$ for some $j \in supp(\mathbf{x}^{u^+})$. Then, there exists a circuit $g \in \mathcal{G}_{in}^{(2)}$ such that $supp(in(g)) \subset supp(m)$. Since $supp(in(g)) \subset supp(f)$, we have

$$supp\left(\frac{LCM(in(f)^p, in(g))}{in(f)^p} \cdot \mathbf{x}^{u^-}\right) = supp(\mathbf{x}^{u^-}) \in \Delta_{in}.$$

Suppose that there exists a monomial m with $supp(m) \cap supp(\mathbf{x}^{u^+}) = \emptyset$ such that $m_1 = m \cdot \prod_{i \in supp(f) \setminus \{j_1\}} x_i \notin \sqrt{in(C_{\mathcal{A}})}$ and $m_2 = m \cdot \prod_{i \in supp(f) \setminus \{j_2\}} x_i \in \sqrt{in(C_{\mathcal{A}})}$ where $j_1, j_2 \in supp(\mathbf{x}^{u^+})$. Then, there exists a circuit $g \in \mathcal{G}_{in}^{(2)}$ such that $supp(in(g)) \subset supp(m_2)$. Thus,

$$supp\left(\frac{LCM(in(f)^p, in(g))}{in(f)^p} \cdot \mathbf{x}^{u^-}\right) \subset supp(m \cdot \mathbf{x}^{u^-}) \subset supp(m_1) \in \Delta_{in},$$

as required. Q. E. D.

If $\mathcal{A}$ is unimodular, then we immediately have the following corollary.

Corollary 3.5. *Suppose that $\mathcal{A}$ is unimodular. Let $in(\cdot)$ be a geometric marking and let $f = \mathbf{x}^{u^+} - \mathbf{x}^{u^-} \in C_{\mathcal{A}}$ with $in(f) = \mathbf{x}^{u^+}$. Then, a triangulation Δ_{in} of $\mathcal{P}_{\mathcal{A}}$ is supported on f if and only if $f \in \mathcal{G}_{in}^{(2)}$ and*

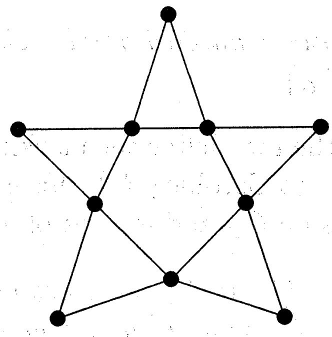
$$\frac{LCM(in(f), in(g))}{in(f)} \cdot \mathbf{x}^{u^-} \in in(C_{\mathcal{A}})$$

for all $g \in \mathcal{G}_{in}^{(2)} \setminus \{f\}$.

4 Some applications

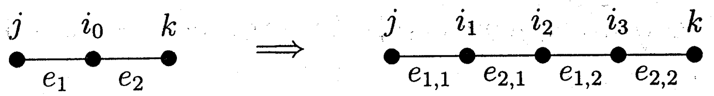
In this section, we study some applications to the problems of polytopes arising from finite graphs. Let G be a finite connected graph having no loop and no multiple edge on the vertex set $V(G) = \{1, 2, \ldots, d\}$ and $E(G) = \{e_1, e_2, \ldots, e_n\}$ the set of edges of G. If $e = \{i, j\}$ is an edge of G joining $i \in V(G)$ with $j \in V(G)$, then we define $\rho(e) \in \mathbb{R}^d$ by $\rho(e) = \mathbf{e}_i + \mathbf{e}_j$. Here $\mathbf{e}_i$ is the i-th unit coordinate vector in $\mathbb{R}^d$. Let $\mathcal{A}_G = \{\rho(e) \; ; \; e \in E(G)\}$. We set $\mathcal{P}_G$ for $\mathcal{P}_{\mathcal{A}_G}$ and call $\mathcal{P}_G$ the *edge polytope* of G. We set $K[G]$ for $K[\mathcal{A}_G]$ and call $K[G]$ the *edge ring* of G and set I_G for $I_{\mathcal{A}_G}$ and call I_G the *toric ideal* of G. See also [O–H$_1$], [O–H$_2$], [O–H$_3$] and [O–H$_4$].

Example 4.1. In [O–H$_1$], we give the following graph G_1 with 10 vertices and 15 edges. Then, $\mathcal{P}_{G_1}$ is a normal (0,1)-polytope none of whose regular triangulations is unimodular and having a unimodular triangulation obtained by one flip from a regular triangulation.



The main purpose of the present section is to give an infinite family of normal edge polytopes having the same property as $\mathcal{P}_{G_1}$.

Suppose that G has a vertex i_0 of degree 2. Then, we can construct a new graph $\widehat{G}$ with $d+2$ vertices and $n+2$ edges by the following operation:



First, by virtue of [O–H$_2$, Corollary 2.3], we have the following proposition.

Proposition 4.2. *Work with the same situation as above. Then, $K[G]$ is normal if and only if $K[\widehat{G}]$ is normal.*

Let $K[\widehat{\mathbf{x}}] = K[x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}, x_3, \ldots, x_n]$ denote the polynomial ring in $n+2$ variables over K. Now, we define the injective homomorphism $\psi : K[\mathbf{x}] \to K[\widehat{\mathbf{x}}]$ by

$$\psi(x_i) = \begin{cases} x_{i,1}x_{i,2} & \text{if } i = 1, 2 \\ & \\ x_i & \text{otherwise.} \end{cases}$$

The set of all circuits of I_G is explicitly classified. Given an even closed walk $\Gamma = (e_{i_1}, e_{i_2}, \ldots, e_{i_{2q}})$ of G with each $e_k \in E(G)$, we write f_Γ for the binomial

$$f_\Gamma = \prod_{k=1}^{q} x_{i_{2k-1}} - \prod_{k=1}^{q} x_{i_{2k}} \in I_G.$$

It is known that I_G is generated by such f_Γ's. See [O–H$_3$, Lemma 1.1]. Let C_G denote the set of all circuits of I_G. Then, the following is known [Stu, Lemma 9.8].

Proposition 4.3. *Let G be a finite connected graph. Then, C_G consists of the binomials f_Γ where Γ is an even closed walk satisfying one of the following conditions:*

(i) Γ *is an even cycle;*

(ii) Γ *consists of two odd cycles having exactly one common vertex;*

(iii) Γ *consists of two odd cycles having no common vertex and connected by a path.*

Corollary 4.4. *Let G be a finite connected graph. Then, the set $C_{\widehat{G}}$ of all circuits equals to $\{\psi(f) \in K[\widehat{\mathbf{x}}] \ ; \ f \in C_G\}$.*

If $in(\cdot)$ is a marking on C_G, then we define the marking $In(\cdot)$ on $C_{\widehat{G}}$ by $In(\psi(f)) = \psi(in(f))$ for all $f \in C_G$. Thanks to Corollary 4.4, this is a one-to-one correspondence between the set of all markings on C_G and the set of all markings on $C_{\widehat{G}}$.

Lemma 4.5. *A marking $in(\cdot)$ on C_G is a geometric marking if and only if corresponding marking $In(\cdot)$ on $C_{\widehat{G}}$ is a geometric marking.*

Proof. Let m be a monomial in $K[\mathbf{x}]$. Note that $m \in \sqrt{in(C_G)}$ if and only if the monomial $\psi(m) \in \sqrt{in(C_{\widehat{G}})}$. Hence, $\psi(\mathcal{G}_{in}^{(i)}) = \mathcal{G}_{In}^{(i)}$ for $i = 1, 2$. It easily follows that there exists a sequence of reductions from m^p to $m' \notin \sqrt{in(C_G)}$ modulo $\mathcal{G}_{in}^{(1)}$ for some positive integer p if and only if there exists a sequence of reductions from $(\psi(m))^p$ to $\psi(m') \notin \sqrt{in(C_{\widehat{G}})}$ modulo $\mathcal{G}_{In}^{(1)}$. By virtue of Theorem 2.6, this completes the proof. Q. E. D.

Lemma 4.6. *Let $in(\cdot)$ and $in'(\cdot)$ be geometric markings on C_G. Then, $\Delta_{in} = \Delta_{in'}$ if and only if $\Delta_{In} = \Delta_{In'}$.*

Proof. Since $\psi(\mathcal{G}_{in}^{(2)}) = \mathcal{G}_{In}^{(2)}$, Corollary 2.5 enable us to complete the proof. Q. E. D.

We define a map $\bar{\psi}$ from the set of all triangulations of $\mathcal{P}_G$ to the set of all triangulations of $\mathcal{P}_{\widehat{G}}$ by $\bar{\psi}(\Delta_{in}) = \Delta_{In}$ where $in(\cdot)$ is a geometric marking on C_G.

Theorem 4.7. *$\bar{\psi}$ is a bijection from the set of all triangulations of $\mathcal{P}_G$ to the set of all triangulations of $\mathcal{P}_{\widehat{G}}$. Moreover, if Δ is a triangulation of $\mathcal{P}_G$, then Δ is regular (resp. unimodular) if and only if $\bar{\psi}(\Delta)$ is regular (resp. unimodular).*

Proof. Thanks to Lemma 4.5 and Lemma 4.6, $\bar{\psi}$ is a bijection from the set of all triangulations of $\mathcal{P}_G$ to the set of all triangulations of $\mathcal{P}_{\widehat{G}}$.

Note that $in(\cdot)$ on C_G is coherent if and only if $In(\cdot)$ on $C_{\widehat{G}}$ is coherent. Thanks to Theorem 2.1, Δ is regular if and only if $\bar{\psi}(\Delta)$ is regular.

Since $I_{\Delta_{in}} = \bigcap_{\sigma \in \Delta_{in}} (x_i \ ; \ \mathbf{a}_i \notin V(\sigma))$, we have $I_{\Delta_{In}} = \bigcap_{\sigma \in \Delta_{in}} (\psi(x_i) \ ; \ \mathbf{a}_i \notin V(\sigma))$. Note that thanks to [O–H2, Lemma 1.4 (i)], either $\mathbf{a}_1 \in V(\sigma)$ or $\mathbf{a}_2 \in V(\sigma)$ for every maximal simplex $\sigma \in \Delta_{in}$. Hence, for each $\sigma \in \Delta_{in}$, we have

$$(\psi(x_i) \ ; \ \mathbf{a}_i \notin V(\sigma)) = \begin{cases} (x_i \ ; \ \mathbf{a}_i \notin V(\sigma)) & \text{if } \mathbf{a}_1, \mathbf{a}_2 \in \sigma \\ \bigcap_{j=1}^{2} \left(\{x_{1,j}\} \cup \{x_i \ ; \ i \neq 1, \mathbf{a}_i \notin V(\sigma)\} \right) & \text{if } \mathbf{a}_1 \notin V(\sigma) \\ \bigcap_{j=1}^{2} \left(\{x_{2,j}\} \cup \{x_i \ ; \ i \neq 2, \mathbf{a}_i \notin V(\sigma)\} \right) & \text{if } \mathbf{a}_2 \notin V(\sigma). \end{cases}$$

It then follows that $\bar{\psi}$ preserves the number of odd cycles in the subgraph of G associated with a maximal simplex $\sigma \in \Delta_{in}$. By virtue of [Stu, Lemma 9.5], this implies that $\bar{\psi}$ preserves the normalized volume of $\sigma \in \Delta_{in}$. Hence, Δ_{in} is unimodular if and only if $\bar{\psi}(\Delta_{in})$ is unimodular as desired. Q. E. D.

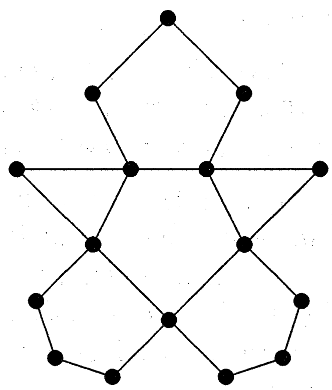
Theorem 4.8. *Let Δ_1 and Δ_2 be triangulations of $\mathcal{P}_G$. Then, Δ_1 is obtained by the flip from Δ_2 along the circuit Γ if and only if $\bar{\psi}(\Delta_1)$ is obtained by the flip from $\bar{\psi}(\Delta_2)$ along the circuit $\psi(\Gamma)$.*

Proof. Let $f, g \in \mathcal{G}_{in}^{(2)}$ and p be a positive integer and let

$$m_1 = \frac{LCM(in(f)^p, in(g))}{in(f)^p} \cdot \mathbf{x}^{u^-} \text{ and } m_2 = \frac{LCM(In(\psi(f))^p, In(\psi(g)))}{In(\psi(f))^p} \cdot \psi(\mathbf{x}^{u^-}).$$

Since $\psi(m_1) = m_2$, $m_1 \in \sqrt{in(C_G)}$ if and only if $m_2 \in \sqrt{In(C_{\widehat{G}})}$. Hence, thanks to Theorem 3.4, Δ_i is supported on Γ if and only if $\bar{\psi}(\Delta_i)$ is supported on $\psi(\Gamma)$. Since $\psi(\mathcal{G}_{in}^{(1)}) = \mathcal{G}_{In}^{(1)}$ and since $supp(in(f)) \subset supp(in(h))$ if and only if $supp(In(\psi(f))) \subset supp(In(\psi(h)))$ for $h \in C_G$, Theorem 3.4 completes the proof. Q. E. D.

From $\mathcal{P}_{G_1}$ in Example 4.1, we get an infinite family of normal edge polytopes having the same property as $\mathcal{P}_{G_1}$ since G_1 has five vertices $\{v_1, v_2, \ldots, v_5\}$ of degree 2. Let $G_{(p_1,p_2,\ldots,p_5)}$ be the graph obtained from G_1 by applying the operation $p_i - 1$ times to the vertex v_i for $1 \leq i \leq 5$. $G_{(p_1,p_2,\ldots,p_5)}$ has $2\sum_{i=1}^{5} p_i$ vertices and $5 + 2\sum_{i=1}^{5} p_i$ edges. For example, $G_{(2,1,2,2,1)}$ is the following graph.



Thanks to Theorem 4.7 and Theorem 4.8, we have the following theorem:

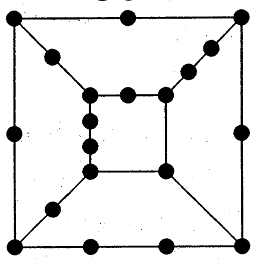
Theorem 4.9. *The edge polytope $\mathcal{P}_{G_{(p_1,p_2,\ldots,p_5)}}$ is a normal (0,1)-polytope none of whose regular triangulations is unimodular and having a unimodular triangulation obtained by one flip from a regular triangulation of $\mathcal{P}_{G_{(p_1,p_2,\ldots,p_5)}}$.*

Finally, we give two examples of graphs whose edge polytope is a normal polytope having a unimodular triangulation and having no regular unimodular triangulation.

Since the graph G_1 given in Example 4.1 has 10 vertices and 15 edges, its edge polytope is of dimension 9 with 15 vertices. Thus, it is reasonable to ask if a graph

G satisfying the *odd cycle condition* [O–H$_2$, Corollary 2.3] has d vertices and n edges with $n - d \leq 4$, then the edge polytope $\mathcal{P}_G$ possesses a regular unimodular triangulation. Note that the operation for graphs defined in this section preserves $n-d$. It is not difficult to show that if a graph G satisfying the odd cycle condition has d vertices and n edges with $n-d \leq 3$, then the edge polytope $\mathcal{P}_G$ possesses a regular unimodular triangulation. Surprisingly, there exists a graph G having 20 vertices and 24 edges whose edge polytope possesses no regular unimodular triangulation.

Example 4.10. Let G_2 be the following graph with 20 vertices and 24 edges.



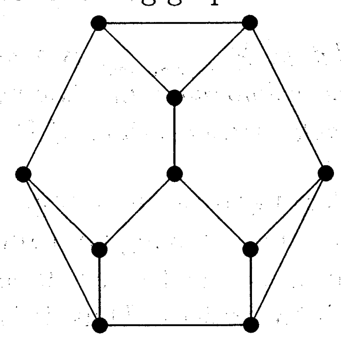
Then, $\dim \mathcal{P}_{G_2} = 19$ and the normalized volume of $\mathcal{P}_{G_2}$ is equal to 307. There are 3 pairs $(C_1, C_1'), (C_2, C_2'), (C_3, C_3')$ of two minimal odd cycles in G_2 having no common vertex. Each (C_i, C_i') has exactly one bridge b_i and the even closed walk $\Gamma_i = (b_i, C_i, b_i, C_i')$ satisfies the conditions in Proposition 4.3. By virtue of [O–H$_3$, Lemma 3.3] or the technique of combinatorial pure subring [O–H–H], we can show that each f_{Γ_i} appears in the reduced Gröbner basis of I_G with respect to any term order. Suppose that I_G has a squarefree initial ideal $in_\succ(I_G)$. Since one of the terms of each f_{Γ_i} is not squarefree, $\succ$ satisfies that

$$\begin{cases} x_1x_3x_4x_6x_8x_{10}x_{11}x_{13}x_{15} \succ x_2x_5x_7x_9x_{12}x_{14}x_{16}x_{19}^2 \\ x_2x_7x_{10}x_{12}x_{17}x_{19}x_{20}x_{22}x_{24} \succ x_1x_3x_6x_{13}x_{18}x_{21}x_{23}x_{11}^2 \\ x_5x_9x_{11}x_{14}x_{16}x_{18}x_{19}x_{21}x_{23} \succ x_4x_8x_{15}x_{17}x_{20}x_{22}x_{24}x_{10}^2 \end{cases}.$$

Since $\prod_{i=1}^3 in_\succ(f_{\Gamma_i}) = \prod_{i=1}^3 (f_{\Gamma_i} - in_\succ(f_{\Gamma_i}))$, this contradicts that $\succ$ is a term order. Thus, with respect to any term order, the initial ideal of I_{G_2} is not squarefree. Thanks to [Stu, Corollary 8.9], $\mathcal{P}_{G_2}$ has no regular unimodular triangulation.

On the other hand, Firla–Ziegler [F–Z] verified that $\mathcal{P}_{G_2}$ *does* have a (nonregular) unimodular triangulation Δ_2. Moreover, H. Imai also verified that Δ_2 is obtained by one flip from a regular triangulation.

Example 4.11. Let G_3 be the following graph with 10 vertices and 15 edges.



By the same technique appearing in Example 4.10, we can see that $\mathcal{P}_{G_3}$ has no regular unimodular triangulation. On the other hand, by explicit computation by PUNTOS, it is verified that $\mathcal{P}_{G_3}$ has a (nonregular) unimodular triangulation Δ_3. $\mathcal{P}_{G_2}$ is the first edge polytope having a unimodular triangulation and none of whose unimodular triangulation is not obtained by one flip from any regular triangulation. However, it is also verified by PUNTOS that Δ_3 is obtained by two flips from a regular triangulation.

We do not know if there exists an edge polytope satisfying odd cycle condition which has no unimodular triangulation so far.

Conjecture 4.12. Let G be a finite connected graph satisfying the odd cycle condition. Then, $\mathcal{P}_G$ has a unimodular triangulation obtained by finite flips from a regular triangulation.

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